Linear Algebra I

18/01/2021, Monday, 15:00 – 18:00

1 Systems of linear equations (2 + 1 + 10 + 2 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are eigenvectors of M corresponding to eigenvalues 1 and 2, respectively. In this problem, we want to find the matrix M.

(a) Find linear equations in the unknowns a, b, c, and d.

(b) Write down the augmented matrix corresponding to the equations found in (a).

(c) Put the augmented matrix into reduced row echelon form.

(d) Find all solutions a, b, c, and d from the reduced row echelon form.

2 Determinants (15 pts)

Let $n \geqslant 1$. Suppose that $A, B \in \mathbb{R}^{n\times n}$ are diagonal matrices. Prove by induction on n that

$$\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = (a_{11}^2 - b_{11}^2)(a_{22}^2 - b_{22}^2)\cdots(a_{nn}^2 - b_{nn}^2).$$

3 Vector spaces (5 + 5 + 5 = 15 pts)

Consider the vector space P_4. The operator $L : P_4 \to P_4$ is given by

$$L(ax^3 + bx^2 + cx + d) := (b + c)x^3 + (c + d)x^2 + (d + a)x + (a + b).$$

(a) Show that L is a linear operator.

(b) Find a basis for $\ker L$.

(c) Let

$$S = \{p \in P_4 \mid p + L(p) = 0\}.$$

Show that S is a subspace of P_4. Find the dimension of S.

4 Matrix representation (10 + 5 = 15 pts)

Consider the functions

$$f_1(x) = \sin x,\ f_2(x) = x\sin x,\ f_3(x) = x^2 \sin x,\ f_4(x) = \cos x,\ f_5(x) = x\cos x,\ f_6(x) = x^2 \cos x$$

and the vector space $V = \text{span}(f_1, f_2, f_3, f_4, f_5, f_6)$. Let $D : V \to V$ be the derivative operator, that is

$$D(f) = f'$$

where f' is the derivative of f.

(a) Find the matrix representation of D relative to the bases $E = F = (f_1, f_2, f_3, f_4, f_5, f_6)$.

(b) Use the matrix representation found in (a) to find the definite integral

$$\int a\,x\cos x + b\,x\sin x\,dx$$

5 Least squares problem (10 + 5 = 15 pts)

Let

$$\boldsymbol{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \boldsymbol{x_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{y_1} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \boldsymbol{y_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

be given vectors. In this problem, we want to find the best least squares fit for

$$\boldsymbol{y} = M\boldsymbol{x}$$

where M is a symmetric matrix of the form

$$M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

(a) Find the normal equations.

(b) Find the least squares solution.

6 Eigenvalues/eigenvectors (4 + 8 + 3 = 15 pts)

Let $A \in \mathbb{R}^{3\times 3}$ be a matrix with the following properties:

(i) sum of the entries on every row is equal to 1.

(ii) $\text{tr}(A) = 1$.

(iii) $\det(A) = -4$.

In this problem, we want to find eigenvalues without finding the characteristic polynomial.

(a) By using (i), show that 1 is an eigenvalue of A.

(b) Find the other two eigenvalues.

(c) Is A diagonalizable? Justify your answer.

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10 pts free